determined for these orientations. Results for other cylinder orientations were presented for several conditions. The accuracy of the approximate expressions was found to be quite good for both pure θ and pure ϕ rotations. Combined rotations were not represented as accurately.

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ANALYTIC SOLUTION FOR THE EIGENVALUES AND COEFFICIENTS OF THE GRAETZ PROBLEM WITH THIRD KIND BOUNDARY CONDITION

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NOMENCLATURE

$C_nR_n(1)$,	coefficient used in equations (4) and (5);
H,	$= hr_w/k$, Biot number;
h,	heat transfer coefficient for external ambient;
h_z,	heat transfer coefficient for flow inside the tube;
k,	thermal conductivity of the fluid;
K,	$= \lambda_n/4$;
Nu_z,	$= 2r_wh_z/k$, Nusselt number;
Pr,	$= \nu/\alpha$, Prandtl number;
r,	$= r'/r_w$, dimensionless radial coordinate;
r',	dimensional radial coordinate;
r_w,	inside radius of the tube;
Re,	$= 2r_wU_m/\nu$, Reynolds number;
T_0, T_∞,	inlet and ambient temperatures respectively;
Um,	mean velocity;
z,	dimensionless axial coordinate, $2z'/r_wRePr$;
z',	axial coordinate.

Greek symbols

θ,	$=(T-T_\infty)/(T_0-T_\infty)$, dimensionless temperature;
λ_n,	eigenvalue.

INTRODUCTION

THE GRAETZ problem for laminar flow inside a circular tube subjected to the boundary condition of the third kind at the tube wall is encountered in numerous engineering applications and has been studied by few investigators [1, 2]. For the analysis of such problems, the local Nusselt number is a quantity of practical interest and its determination requires a knowledge of the eigenvalues and the eigenfunctions for the problem. Hsu [2] solved such an eigenvalue problem numerically and also presented some asymptotic expressions for the eigenvalues and the coefficients. Here we present highly accurate analytic expressions for the determination of the eigenvalues and the coefficients that are applicable over the entire range of the Biot number from zero to infinity.

ANALYSIS

We consider thermally developing laminar flow inside a circular tube with fully developed velocity profile and subjected to the boundary condition of the third kind at the tube wall. For a constant property, incompressible fluid with no heat generation and neglecting the viscous energy dissipation, the mathematical formulation of this heat transfer problem is given in the dimensionless form as

$$2(1-r^2)\frac{\partial\theta(r,z)}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right), \quad 0 \le r \le 1, \quad z > 0 \tag{1a}$$

$$\frac{\partial\theta(0,z)}{\partial r} = 0, \quad z > 0 \tag{1b}$$

$$\frac{\partial\theta(1,z)}{\partial r} + H\theta(1,z) = 0, \quad z > 0 \tag{1c}$$

$$\theta(r,0) = 1, \quad 0 \le r \le 1. \tag{1d}$$

The solution of this heat transfer problem is given by

$$\theta(r,z) = \sum_{n=0}^{\infty} C_nR_n(r)\mathrm{e}^{-\lambda_n^2 z/2} \tag{2}$$

where λ_n and $R_n(r)$ are the eigenvalues and the eigenfunctions of the following eigenvalue problem

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\lambda^2(1-r^2)R(r)=0,\quad 0\le r\le 1 \tag{3a}$$

$$\frac{dR(0)}{dr}=0, \tag{3b}$$

$$\frac{dR(1)}{dr}+HR(1)=0. \tag{3c}$$

It can be readily shown that the local Nusselt number, Nu_z is given by

$$Nu_z\equiv\frac{2r_wh_z}{k}=\frac{2\sum_{n=0}^{\infty}C_nR_n(1)e^{-\lambda_n^2z/2}}{\sum_{n=0}^{\infty}\left(\frac{4}{\lambda_n^2}-\frac{1}{H}\right)C_nR_n(1)e^{-\lambda_n^2z/2}}. \tag{4}$$

For the case of constant wall heat flux, q_w, the boundary condition (1c) in the above heat transfer problem should be modified accordingly; but the eigenvalue problem (3) is applicable by setting $H=0$. The local Nusselt number for the case of constant wall heat flux takes the form

$$Nu_z\equiv\frac{2r_wh_z}{k}=\frac{1}{\frac{11}{48}+\frac{1}{2}\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2z/2}}. \tag{5}$$

Clearly, the local Nusselt number can readily be computed from equations (4) and (5) if the eigenvalues λ_n and the coefficients $C_nR_n(1)$ are available for a given value of the Biot number, H, including the well known special case of $H=0$. However, one needs to solve the eigenvalue problem (3) to determine the eigenvalues and the eigenfunctions. Although such problems can be solved numerically [2], the determination of the higher eigenvalues by purely numerical scheme becomes very difficult. Therefore, some asymptotic expressions have also been presented in [2].

Here we present highly accurate asymptotic expressions for the determination of the eigenvalues, λ_n, and the coefficients, $C_nR_n(1)$, which are valid over the entire range of Biot number. We used the matched asymptotic expansion technique to obtain the solutions. Such a technique has been used previously by, for example, Shibani and Özişik [3] to develop asymptotic expressions for the eigenvalues associated with the thermal entry region heat transfer subjected to the boundary condition of the first kind.

The eigenvalues λ_n can be determined from

$$\lambda_n=4K \tag{6}$$

where K is the solution of the transcendental equation

$$K=n+\frac{1}{2}\left(1+\frac{1}{3}\frac{\omega-1}{\omega+1}\right)+\frac{(-1)^n}{2\pi K}[0.1237871+\beta\ln(\pi K)]\left[\frac{\omega\cos\varepsilon_1+\cos\varepsilon_2}{\omega+1}\right], \tag{7a}$$

$$\omega=0.4319535\frac{H-0.5}{K^{2/3}}, \tag{7b}$$

$$\varepsilon_1=\pi K-\frac{2\pi}{3}, \tag{7c}$$

$$\varepsilon_2=\pi K-\frac{\pi}{3}. \tag{7d}$$

For the special case of $H\to\infty$ (i.e. isothermal wall), equation (7b) reduces to $(1/\omega)\to 0$, and for higher values of K's equation (7c) is approximated by $\varepsilon_1\simeq n\pi$ and equation (7a) simplifies to

$$K=n+\frac{2}{3}+\frac{1}{2\pi K}[0.1237871+\beta\ln(\pi K)]\quad\text{for } H\to\infty. \tag{8}$$

The coefficients $C_nR_n(1)$ appearing in equations (4) and (5) can be determined from

$$C_nR_n(1)=\frac{A}{K^{5/3}}\frac{\frac{1}{2K}[0.1237871+\beta\ln(\pi K)]\cos\varepsilon_1-\sin\varepsilon_1}{\cos\varepsilon_2+\omega\cos\varepsilon_1+\frac{B}{2K}+\frac{D}{2\pi K^2}}\quad\text{for } H<10 \tag{9a}$$

where

$$B=[-0.1945227+\beta\ln(\pi K)](\sin\varepsilon_2+\omega\sin\varepsilon_1)+0.4244131\sin\varepsilon_2, \tag{9b}$$

$$D=\{-\beta+\tfrac{3}{2}[0.1237871+\beta\ln(\pi K)]\}(\cos\varepsilon_2+\omega\cos\varepsilon_1)-\tfrac{2}{3}[0.1237871+\beta\ln(\pi K)]\cos\varepsilon_2, \tag{9c}$$

$$A=\begin{cases}-0.27499 & \text{for } H=0\\ +0.27499H & \text{for } H\neq 0\end{cases} \tag{9d}$$

and from

$$C_nR_n(1)=\frac{2}{\pi K}\frac{(\sin\varepsilon_2-\alpha\sin\varepsilon_1)-\frac{1}{2K}[0.1237871+\beta\ln(\pi K)](\cos\varepsilon_2-\alpha\cos\varepsilon_1)}{\cos\varepsilon_2+\omega\cos\varepsilon_1+\frac{B}{2K}+\frac{D}{2\pi K^2}}\quad\text{for } H\ge 10 \tag{10a}$$

where

$$\alpha=\frac{0.2159767}{K^{2/3}} \tag{10b}$$

and B and D are defined by equations (9b) and (9c) respectively.

In the foregoing relations, the values of β are determined by equating the last (i.e. 12th) exact eigenvalue or the coefficient, $C_nR_n(1)$, with the asymptotic ones. We present in Table 1 the values of β, for different values of H, determined by equating the eigenvalues for lower values of H's (i.e. $H<10$), and equating the coefficients for the higher values of H (i.e. $H\ge 10$). This procedure yielded better results.

In equations (9d) and (10a) one can even neglect the term in the denominator that contains $(1/K^2)$ with little effect in the results.

Table 1. Values of β

H	β	H	β
0	− 0.026597	5	− 0.200748
0.1	− 0.039567	10	− 0.085227
0.5	− 0.082274	50	− 0.067512
1.0	− 0.124408	100	− 0.055986
2.0	− 0.179048		

RESULTS

To illustrate the accuracy of the foregoing expressions for the determination of the eigenvalues λ_n and the coefficients, $C_n R_n(1)$, we present in Table 2 a comparison of the results obtained from our asymptotic expressions (6), (9) and (10), with those obtained from the exact solution of the eigenvalue problem (3) by employing the procedure described in [4]. Also included in this table are the results calculated from the asymptotic expressions given by Hsu [2] and [5] for the boundary conditions of the third and second kind at the wall. Clearly, the analytical expressions given in this paper can predict the eigenvalues and the coefficients over the entire spectrum of eigenvalues for all values of the Biot number very close to the exact results.

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Table 2. The eigenvalues and coefficients

H	n	λ_n			$C_n R_n(1)$		
		Exact	Present	Hsu [5]	Exact	Present	Hsu [5]
0	1	5.06750	4.97681	5.33333	-0.19872	-0.21091	-0.14748
	2	9.15761	9.12198	9.33333	-0.06926	-0.07034	-0.05803
	3	13.19722	13.17660	13.33333	-0.03652	-0.03674	-0.03203
	4	17.22021	17.20631	17.33333	-0.02301	-0.02306	-0.02068
	5	21.23550	21.22533	21.33333	-0.01603	-0.01604	-0.01463
	6	25.24652	25.23866	25.33333	-0.01191	-0.01190	-0.01099
	7	29.25490	29.24861	29.33333	-0.00925	-0.00924	-0.00861
	8	33.26151	33.25635	33.33333	-0.00743	-0.00741	-0.00695
	9	37.26689	37.26256	37.33333	-0.00612	-0.00611	-0.00576
	10	41.27135	41.26768	41.33333	-0.00514	-0.00513	-0.00486
	11	45.27510	45.27199	45.33333	-0.00439	-0.00438	-0.00417
	12	49.27847	49.27565	49.33333	-0.00380	-0.00379	-0.00362
		λ_n			$C_n R_n(1)$		
H	n	Exact	Present	Hsu [2]	Exact	Present	Hsu [2]
0.1	0	.61834	.73760	1.42412	0.95553	0.49904	0.11708
	1	5.11687	5.14232	5.37178	0.01891	0.01849	0.01399
	2	9.18892	9.19991	9.36003	0.00673	0.00664	0.00562
	3	13.22109	13.22705	13.35445	0.00358	0.00354	0.00313
	4	17.23988	17.24344	17.35109	0.00226	0.00224	0.00203
	5	21.25241	21.25465	21.34882	0.00158	0.00157	0.00144
	6	25.26146	25.26288	25.34715	0.00117	0.00117	0.00108
	7	29.26834	29.26924	29.34587	0.00091	0.00091	0.00085
	8	33.27379	33.27432	33.34485	0.00073	0.00073	0.00069
	9	37.27824	37.27850	37.34401	0.00060	0.00060	0.00057
	10	41.28191	41.28201	41.34332	0.00051	0.00051	0.00048
	11	45.28502	45.28500	45.34275	0.00043	0.00043	0.00041

Table 2 (continued)

H	n	λ_n			$C_nR_n(1)$		
		Exact	Present	Hsu [2]	Exact	Present	Hsu [2]
1.0	0	1.64125	1.79492	1.87832	0.66034	0.47276	0.33502
	1	5.47831	5.50623	5.64969	0.11949	0.11149	0.08899
	2	9.43597	9.44593	9.56830	0.05111	0.04904	0.04223
	3	13.41524	13.41973	13.52486	0.02915	0.02826	0.02524
	4	17.40259	17.40482	17.49722	0.01917	0.01868	0.01705
	5	21.39392	21.39507	21.47784	0.01372	0.01341	0.01241
	6	25.38754	25.38812	25.46336	0.01038	0.01018	0.00951
	7	29.38261	29.38289	29.45206	0.00818	0.00804	0.00757
	8	33.37868	33.37878	33.44295	0.00664	0.00653	0.00619
	9	37.37544	37.37546	37.43541	0.00552	0.00544	0.00517
	10	41.37270	41.37271	41.42905	0.00467	0.00461	0.00440
	11	45.37039	45.37038	45.42360	0.00402	0.00396	0.00379
10	0	2.51675	2.52544	2.49587	0.13764	0.12182	0.11972
	1	6.36460	6.33044	6.37146	0.08426	0.08049	0.07526
	2	10.27069	10.23724	10.28502	0.006218	0.06001	0.05631
	3	14.20020	14.17171	14.21802	0.04925	0.04778	0.04501
	4	18.14366	18.12051	18.16339	0.04057	0.03954	0.03735
	5	22.09662	22.07846	22.11749	0.03431	0.03358	0.03177
	6	26.05653	26.04285	26.07809	0.02956	0.02906	0.02751
	7	30.02179	30.01205	30.04373	0.02585	0.02552	0.02416
	8	33.99124	33.98500	34.01340	0.02287	0.02267	0.02145
	9	37.96411	37.96095	37.98633	0.02042	0.02032	0.01922
	10	41.93976	41.93935	41.96199	0.01839	0.01837	0.01735
	11	45.91780	45.91978	45.93993	0.01667	0.01671	0.01578
100	0	2.68427	2.71850	2.64745	0.01486	0.01370	0.01366
	1	6.64321	6.63641	6.63149	0.01064	0.01038	0.01042
	2	10.62493	10.61094	10.61882	0.00894	0.00881	0.00883
	3	14.61161	14.59481	14.60780	0.00794	0.00784	0.00785
	4	18.60043	18.58205	18.59785	0.00724	0.00716	0.00717
	5	22.59052	22.57103	22.58868	0.00672	0.00665	0.00665
	6	26.58147	26.56111	26.58012	0.00631	0.00624	0.00624
	7	30.57301	30.55196	30.57207	0.00597	0.00590	0.00590
	8	34.56508	34.54341	34.56441	0.00568	0.00562	0.00562
	9	38.55756	38.53534	38.55710	0.00543	0.00537	0.00537
	10	42.55040	42.52768	42.55009	0.00521	0.00515	0.00515
	11	46.54349	46.52036	46.54335	0.00502	0.00496	0.00496